Ben Talbot

Calc 323 11/29/21

<u>§16.? Surface Integrals</u>

Last time: $\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) |\vec{S}_u \times \vec{S}_v|\,dA$

where $\vec{S}(u,v)$ parameterizes S on domain D.

<u>Ex:</u> Compute $\iint_S x^2\,dS$ for S, the unit sphere @origin.

<u>Sol:</u> First, we parameterize the unit sphere by

$$\vec{S}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

spherical coords

$$\text{on } (\theta,\phi) \in [0, 2\pi] \times [0, \pi]$$

$$\vec{S}_\theta = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0 \rangle$$
$$= \sin(\phi) \langle -\sin(\theta), \cos(\theta), 0 \rangle$$

$$\vec{S}_\phi = \langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi) \rangle$$

$$\therefore \vec{S}_\theta \times \vec{S}_\phi = \sin(\phi) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix}$$

$$= \sin(\phi) \langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi)\sin^2(\theta) - \cos(\phi)\cos^2(\theta) \rangle$$

$$= \sin(\phi) \langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi) \rangle$$

$$\therefore \iint_S x^2\,dS = \iint_D \sin^2(\phi)\cos^2(\theta) |\sin(\phi) \langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi) \rangle|\,dA$$

$$= \iint_D \sin^3(\phi)\cos^2(\theta) \sqrt{\sin^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos^2(\phi)}\,dA$$

$$= \iint_D \sin^3(\phi)\cos^2(\theta)\,dA =$$

$\longrightarrow$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin(\phi)(1-\cos^2(\phi)) \cdot \frac{1}{2}(1+\cos(2\theta))\, d\phi\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta)) \int_{u=1}^{-1} -(1-u^2)\, du\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta)) \left[u - \frac{1}{3}u^3\right]_{u=1}^{-1} d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta))\left(\left(-1+\frac{1}{3}\right) - \left(1-\frac{1}{3}\right)\right) d\theta$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_{\theta=0}^{2\pi} (1+\cos(2\theta))\, d\theta$$

$$= \frac{2}{3}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} = \frac{2}{3}(2\pi - 0) = \boxed{\frac{4}{3}\pi.}$$

<u>GOAL</u>: Build a theory of surface integrals for vector fields (analogous to line integrals)

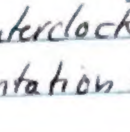
<u>BUT</u>: We need to think about "orientation" for surfaces first...

("direction") ↳ should involve the right-hand rule.

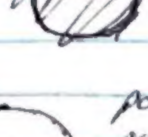
For line integrals:

opposite direction negates the line integral result...

<u>A</u>: Orientation means (for surfaces) "consistent choice of normal to the tangent planes".

<u>Ex</u>:  pos. orientation: OUTWARD. ∴ neg. orientation: INWARD.

alternatively, we want "counterclockwise from above" orientation to be positive



pos. orientation on torus is open outward.

Q: Can we do this for every surface?

Choose consistent pos. orientation.
Mobius strip → non-orientable...

NB: Our theory of surface integrals chokes on non-orientable surfaces... From here on, the surfaces we work with are orientable.

NB2: Choosing a parametrization of S by $\vec{S}(u,v)$ automatically chooses an orientation: $\vec{n}(u,v) = \frac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}$

Defn: The flux of vector field $\vec{v}$ across surface S is $\iint_S \vec{v} \cdot d\vec{S} = \iint_S \vec{v} \cdot \vec{n}\, dS$

$$= \iint_D \vec{v}(u,v) \cdot \frac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}\, dA$$

$$= \iint_D \vec{v} \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

where $\vec{S}(u,v)$ is a parameterization of S on domain D.

Ex: Compute the flux of $\vec{v} = \langle z, y, x \rangle$ across the sphere of radius 1, centered @ origin.

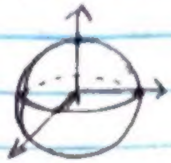
~~Sol~~ Convention: if orientation not explicitly given, it is implicitly the positive orientation.

Sol: As before, $\vec{S}(\theta, \phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$ on $D = [0, 2\pi] \times [0, \pi]$ and

$$\vec{S}_\theta \times \vec{S}_\phi = -\sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

Check: outward orientation?

At a test point, the ~~north~~ east pole, ~~(0,0,1)~~ (1,0,0)
i.e. $(\theta, \phi) = (0, \frac{\pi}{2})$



$$\therefore \ (\vec{S}_\theta \times \vec{S}_\phi)(0, \tfrac{\pi}{2}) = -\langle 1,0,0\rangle = \langle -1, 0, 0\rangle$$

(inward orientation)

$\therefore$ we need to use $-(\vec{S}_\theta \times \vec{S}_\phi)$ instead.

now, $\vec{v}(\theta, \phi) \cdot (\vec{S}_\theta, \vec{S}_\phi)$

$$= \langle \cos\phi, \ \sin\phi\sin\theta, \ \sin\phi\cos\theta \rangle$$

$$\cdot -\sin\phi \langle \sin\phi\cos\theta, \ \sin\phi\sin\theta, \ \cos\phi \rangle$$

$$= -\sin\phi \, (2\cos\phi\sin\phi\cos\theta + \sin^2\phi\sin^2\theta) =$$

$$\therefore \iint_S \vec{v} \cdot d\vec{S} = \iint_D \vec{v} \cdot (-\vec{S}_\phi \times \vec{S}_\theta) \, dA$$

$$= \iint_D \sin(\phi)(2\cos\phi\sin\phi\cos\theta + \sin^2\phi\sin^2\theta) \, dA$$

$$= 2\iint_D \cos\phi\sin^2\phi\cos\theta \, dA + \iint_D \sin^3\phi\sin^2\theta \, dA$$

Now $\iint_D \cos\phi\sin^2\phi\cos\theta \, dA$

$$= \int_{\phi=0}^{\pi} \cos\phi\sin^2\phi \int_{\theta=0}^{2\pi} \cos\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi} \cos\phi\sin^3\phi \cdot 0 \, d\phi = \boxed{0}$$